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On Various Types of Ideals of Gamma Rings and the Corresponding Operator Rings

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Abstract
The prime objective of this paper is to prove some deep results on various types of Gamma ideals. The characteristics of various types of Gamma ideals viz. prime/maximal/minimal/nilpotent/primary/semi-prime ideals of a Gamma-ring are shown to be maintained in the corresponding right (left) operator rings of the Gamma-rings. The converse problems are also investigated with some good outcomes. Further it is shown that the projective product of two Gamma-rings cannot be simple.
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I. Introduction

Ideals are the backbone of the Gamma-ring theory. Nobusawa developed the notion of a Gamma-ring which is more general than a ring [3]. He obtained the analogue of the Wedderburn theorem for simple Gamma-ring with minimum condition on one-sided ideals. Barnes [6] weakened slightly the defining conditions for a Gamma-ring, introduced the notion of prime ideals, primary ideals and radical for a Gamma-ring, and obtained analogues of the classical Noether-Lasker theorems concerning primary representations of ideals for Gamma-rings. Many prominent mathematicians have extended fruitfully many significant technical results on ideals of general rings to those of Gamma-rings [1,2,4,7,9].

II. Basic Concepts

Definition 2.1: A gamma ring (X, Γ) in the sense of Nabusawa is said to be **simple** if for any two nonzero elements $x, y \in X$, there exist $\gamma \in \Gamma$ such that $x\gamma y \neq 0$.

Definition 2.2: If I is an additive subgroup of a gamma ring (X, Γ) and $X \Gamma I \subseteq I$ (or $I \Gamma X \subseteq$I), then I is called a left (or right) gamma ideal of X. If I is both left and right gamma ideal then it is said to be a gamma ideal of (X, Γ) or simply an ideal.

Definition 2.3: An ideal I of a gamma ring (X, Γ) is said to be **prime** if for any two ideals A and B of X, $A \Gamma B \subseteq I \Rightarrow A \subseteq I \text{ or } B \subseteq I$. I is said to be **semi-prime** if for any ideal U of X, $U \Gamma U \subseteq I \Rightarrow U \subseteq I$.

Definition 2.4: A nonzero right (or left) ideal I of a gamma ring (X, Γ) is said to be a **minimal** right (or left) ideal if the only right (or left) ideal of X contained in I are 0 and I itself.

Definition 2.5: A nonzero ideal I of a gamma ring (X, Γ) such that $I \neq X$ is said to be **maximal** ideal, if there exists no proper ideal of X containing I.

Definition 2.6: An ideal I of a gamma ring (X, Γ) is said to be **primary** if it satisfies,
$a\gamma b \subseteq I, a \nsubseteq I \Rightarrow b \subseteq J \ \forall\ a, b \in X \text{ and } \gamma \in \Gamma$.
Where $J = \{x \in X: (x\gamma)^{n-1}x \in I \text{ for some } n \in N \text{ and } \gamma \in \Gamma\}$
and $(x\gamma)^{n-1}x = x \text{ when } n = 1$.

Definition 2.7: An ideal I of a gamma ring (X, Γ) is said to be **nilpotent** if for some positive integer n, $I^n = 0$. Where we denote I^n by the set $I \Gamma I \Gamma \ldots\ldots \Gamma I$ (all finite sums of the form $\sum x_1\gamma_1 x_2\gamma_2 \ldots\ldots \gamma_{n-1}x_n$ with $x_i \in I$ and $\gamma_i \in \Gamma$).

Definition 2.8: Let (X_1, Γ_1) and (X_2, Γ_2) be two gamma rings. Let $X = X_1 \times X_2$ and $\Gamma = \Gamma_1 \times \Gamma_2$. Then defining addition and multiplication on X and Γ by,
$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$,
$(\alpha_1, \alpha_2) + (\beta_1, \beta_2) = (\alpha_1 + \beta_1, \alpha_2 + \beta_2)$
and $(x_1, x_2)(\alpha_1, \alpha_2)(y_1, y_2) = (x_1\alpha_1 y_1, x_2\alpha_2 y_2)$
for every $(x_1, x_2), (y_1, y_2) \in X$ and $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in \Gamma$,
(X, Γ) is a gamma ring. We call this gamma ring as the **Projective product of gamma rings**.

III. Main Results

Theorem 3.1: The Projective product of two gamma rings (X_1, Γ_1) and (X_2, Γ_2) can never be a simple gamma ring.

Proof: Let (X, Γ) be the projective product of the gamma rings (X_1, Γ_1) and (X_2, Γ_2),
Let $x = (x_1, 0), y = (0, y_2) \in X$ be two nonzero elements. Then $x_1 \neq 0, y_2 \neq 0$
Then for any $\alpha = (\alpha_1, \alpha_2) \in \Gamma$, we get
$x\alpha y = (x_1, 0)(\alpha_1, \alpha_2)(0, y_2) = (x_1\alpha_1 0, 0\alpha_2 y_2)$
$= (0,0) = 0$
Thus for these nonzero $x, y \in X$, there does not exist any $\alpha \in \Gamma$ such that $x\alpha y \neq 0$.
Thus (X, Γ) is not a simple gamma ring and hence the result.
If R and L are the right and left operator rings respectively of the gamma ring (X, Γ), then the forms of R and L are
$R = \{\sum_i [\gamma_i, x_i]: \gamma_i \in \Gamma, x_i \in X\}$ and $L = \{\sum_i [x_i, \gamma_i]: \gamma_i \in \Gamma, x_i \in X\}$
Then defining a multiplication in R by, $\sum_i [\alpha_i, x_i] \sum_j [\beta_j, y_j] = \sum_{i,j} [\alpha_i, x_i\beta_j y_j]$, it can be verified that R forms a ring with ordinary addition of endomorphisms and the above defined multiplication. Similar verification can also be done on the left operator ring L with a defined multiplication. In this paper we shall discuss various types of ideals in a gamma ring (X, Γ) and their corresponding ideals in the right operator ring R.

Theorem 3.2: Every left (or right) ideal of (X, Γ) defines a left (or right) ideal of the right operator ring R and conversely.

Proof: Let I be a left ideal of a gamma ring (X, Γ). We define a subset R' of R by,
$R' = \{\sum_i [\gamma_i, x_i]: \gamma_i \in \Gamma, x_i \in I\}$. We show R' is a left ideal of R.
For this let $x = \sum_i [\gamma_i, x_i] \in R'$ and $r = \sum_j [\alpha_j, a_j] \in R$ be any elements, where $\gamma_i, \alpha_j \in \Gamma$; $x_i \in I$; $a_j \in X$ for i, j.
Now, $rx = \sum_j [\alpha_j, a_j] \sum_i [\gamma_i, x_i] = \sum_{i,j} [\alpha_j, a_j\gamma_i x_i]$
Since $x_i \in I$; $a_j \in X$; $\gamma_i \in \Gamma$ for i, j and I is a left ideal of X, so
$a_j\gamma_i x_i \in I => \sum_{i,j} [\alpha_j, a_j\gamma_i x_i] \in R' => rx \in R'$ for all $r \in R$ and $x \in R'$
So R' is a left ideal of the right operator ring R.

Conversely, let P be a left ideal of R. Then P will be of the form
$P = \{\sum_i [\alpha_i, x_i]: \alpha_i \in \Gamma, x_i \in J \subseteq X\}$. We show J is a left ideal of X.
Let $x \in X, a \in J$ be any two elements. Then for any $\in \Gamma$, $[\gamma, x] \in R, [\gamma, a] \in P$.
Since P is a left ideal of R, so, $[\gamma, x][\gamma, a] \in P =>$ $[\gamma, x\gamma a] \in P => x\gamma a \in J$
So J is a left ideal of X and hence the result.
This result can similarly be proved for right ideals also. Thus every ideal in a gamma ring defines an ideal in the right operator ring.

Theorem 3.3: Every prime ideal of (X, Γ) defines a prime ideal of the right operator ring R and conversely.

Proof: Let I be a prime ideal of a gamma ring (X, Γ). We define a subset R' of R by,
$R' = \{\sum_i [\gamma_i, x_i]: \gamma_i \in \Gamma, x_i \in I\}$. We show R' is a prime ideal of R.
By Result 3.2, R' is an ideal of the right operator ring R. We just need to show the prime part. For this let, $xy \in R'$, where $x, y \in R$.
Then $x = \sum_i [\alpha_i, x_i]$, $y = \sum_j [\beta_j, y_j]$ where $\alpha_i, \beta_j \in \Gamma$ and $x_i, y_j \in I$
Now, $xy = \sum_i [\alpha_i, x_i] \sum_j [\beta_j, y_j] \in R'$
$=> xy = \sum_{i,j} [\alpha_i, x_i\beta_j y_j] \in R'$
$=> x_i\beta_j y_j \in I$ for i, j
$=> x_i \in I$ or $y_j \in I$ for i, j [Since I is a prime ideal of X]
If $x_i \in I$ for i, then $\sum_i [\alpha_i, x_i] \in R'$ for $\alpha_i \in \Gamma$ which implies that $x \in R'$.
Again If $y_j \in I$ for j, then $\sum_j [\beta_j, y_j] \in R'$ for $\beta_j \in \Gamma$ which implies that $y \in R'$.
Thus, $xy \in R'$ implies $x \in R'$ or $y \in R'$. So R' is a prime ideal of the right operator ring R.
Conversely, let P be a prime ideal of R. Then P will be of the form
$P = \{\sum_i [\gamma_i, a_i]: \gamma_i \in \Gamma, a_i \in J \subseteq X\}$. We show J is a prime ideal of X.
Let $a, b \in X$ be any two elements such that $a\gamma b \in J$ for $\gamma \in \Gamma$.
Since $a, b \in X$ and $\gamma \in \Gamma$, so $[\gamma, a], [\gamma, b] \in R$
Again, $a\gamma b \in J => [\gamma, a\gamma b] \in P$
$=> [\gamma, a][\gamma, b] \in P$
$=> [\gamma, a] \in P$ or $[\gamma, b] \in P$ [Since P is a prime ideal of R]
If $[\gamma, a] \in P$ then $a \in J$ and if $[\gamma, b] \in P$ then $b \in J$.
Thus $a\gamma b \in J => a \in J$ or $b \in J$. So J is a prime ideal of X and hence the result.

Theorem 3.4: Every minimal ideal of (X, Γ) defines a minimal ideal of the right operator ring R and conversely.

Proof: Let I be a minimal ideal of a gamma ring (X, Γ). We define a subset R' of R by,

$R' = \{\sum_i[\gamma_i, x_i]: \gamma_i \in \Gamma, x_i \in I\}$. We show R' is a minimal ideal of R.
By Result 3.2, R' is an ideal of the right operator ring R. We just need to show the minimal part.
Suppose R' is not minimal. Then there exists an ideal A of R in between 0 and R' i.e $A \neq 0$, $A \subseteq R'$ but $A \neq R'$.
Let $A = \{\sum_j[\alpha_j, y_j]: \alpha_j \in \Gamma, y_j \in J \subseteq X\}$. Since A is an ideal of R so by result 3.2, J is also an ideal of X.
Since $A \neq R'$, so there exists an element $x = \sum_i[\gamma_i, x_i] \in R'$ but $x \notin A$.
$\Rightarrow x_i \in I$ but $x_i \notin J \Rightarrow J \neq I$.
Again since $A \neq R'$, so obviously $J \subseteq I$. Also $A \neq 0$, so $J \neq 0$.
Thus there exists an ideal J of X in between 0 and I, which contradicts the fact that I is a minimal ideal of X, i.e our supposition was wrong. So R' is a minimal ideal of R.
Conversely, let $P = \{\sum_i[\gamma_i, x_i]: \gamma_i \in \Gamma, x_i \in I\}$ be a minimal ideal of R. Then I is an ideal of X.
We show I is minimal. Suppose not. Then $\exists$ an ideal J of X in between 0 and I i.e $0 \subsetneq J \subsetneq I$.
We define a subset Q of R by $Q = \{\sum_j[\alpha_j, y_j]: \alpha_j \in \Gamma, y_j \in J\}$. Then Q is an ideal of R.
Since $J \neq I$ and $J \subseteq I$, so there exists an element $x \in I$ but $x \notin J$.
Then for any $\gamma \in \Gamma$, $[\gamma, x] \in P$ *but* $[\gamma, x] \notin Q \Rightarrow P \neq Q$.
Again since $J \subseteq I \Rightarrow Q \subseteq P$ and $J \neq 0 \Rightarrow Q \neq 0$.
Thus there exists an ideal Q of R which lies in between 0 and P, which contradicts that P is a minimal ideal of R. Thus I is a minimal ideal of X and hence the result.

Theorem 3.5: Every maximal ideal of (X, Γ) defines a maximal ideal of the right operator ring R and conversely.

Proof: Let I be a maximal ideal of a gamma ring (X, Γ). We define a subset R' of R by,
$R' = \{\sum_i[\gamma_i, x_i]: \gamma_i \in \Gamma, x_i \in I\}$. We show R' is a maximal ideal of R. Since I is maximal so I is nonzero and hence R' is also nonzero.
By Result 3.2, R' is an ideal of the right operator ring R. We just need to show the maximal part.
On the contrary, if possible let R' be not maximal. Then there exists a proper ideal P of R containing R' i.e $R' \subseteq P \subseteq R$. Let $P = \{\sum_j[\alpha_j, y_j]: \alpha_j \in \Gamma, y_j \in J \subseteq X\}$. Then J is an ideal of X.
Let $x \in I$ be any element.
$\Rightarrow [\gamma, x] \in R'$ for all $\gamma \in \Gamma \Rightarrow [\gamma, x] \in P$ for all $\gamma \in \Gamma \Rightarrow x \in J \Rightarrow I \subseteq J$
Again since P is a proper ideal of R so J is also a proper ideal of X.

Thus we get a proper ideal J of X containing I, which contradicts that I is a maximal idal of X. So R' is a maximal ideal of the right operator ring R.

Conversely, let $M = \{\sum_i[\gamma_i, x_i]: \gamma_i \in \Gamma, x_i \in A\}$ be a maximal ideal of R. Then obviously A is an ideal of X. We show A is maximal.

Since M is maximal, so M is nonzero and there does not exist any proper ideal of R containing M. Since M is nonzero so obviously A is also nonzero.
On the contrary, let A be not maximal. Then $\exists$ a proper ideal B of X containing A i.e $A \subseteq B \subsetneq X$.
We construct a subset $N = \{\sum_j[\alpha_j, y_j]: \alpha_j \in \Gamma, y_j \in B\}$ of R. Since $B \subsetneq X \Rightarrow N \subsetneq R$.
Let $m = \sum_i[\gamma_i, x_i] \in M$ be any element.
Then $\gamma_i \in \Gamma$ and $x_i \in A$ for i
Since $A \subseteq B$ so $x_i \in A \Rightarrow x_i \in B \Rightarrow \sum_i[\gamma_i, x_i] \in N \Rightarrow m \in N$
So, $M \subseteq N$. Thus we found a proper ideal N of R containing M, which contradicts that M is a maximal ideal of R. So A is maximal. Hence the result.

Theorem 3.6: Every nilpotent ideal of (X, Γ) defines a nilpotent ideal of the right operator ring R and conversely.

Proof: Let I be a nilpotent ideal of a gamma ring (X, Γ). We define a subset P of R by,
$P = \{\sum_i[\gamma_i, x_i]: \gamma_i \in \Gamma, x_i \in I\}$, which is an ideal of the right operator ring R. We show P is a nilpotent ideal of R.
Since I is nilpotent, so there exists a positive integer n such that $I^n = 0$ i..e $I \Gamma I \Gamma \ldots\ldots \Gamma I = 0$.
i.e all elements of the form $\sum x_1\gamma_1x_2\gamma_2 \ldots\ldots \gamma_{n-1}x_n$ are zero, where $x_i \in I$ and $\gamma_i \in \Gamma$. ….(1)
Let $x \in P^n$ be any element. Then x will be of the form,
$x = a_1a_2 \ldots\ldots a_n$ where $a_j = \sum_i[\gamma_{i_j}, x_{i_j}] \in P$.

$$\text{Then } x = \sum_i[\gamma_{i_1}, x_{i_1}]\sum_i[\gamma_{i_2}, x_{i_2}] \ldots\ldots \sum_i[\gamma_{i_n}, x_{i_n}]$$
$$= \sum_i[\gamma_{i_1}, x_{i_1}\gamma_{i_2}x_{i_2} \ldots\ldots \gamma_{i_n}x_{i_n}]$$
$$= \sum_i[\gamma_{i_1}, 0] \quad \text{[Using (1)]}$$
$$= 0$$

So, $x \in P^n \Rightarrow x = 0$ which implies $P^n = 0$. So P is a nilpotent ideal of R.

Conversely, let $P = \{\sum_i[\gamma_i, x_i]: \gamma_i \in \Gamma, x_i \in I\}$ be a nilpotent ideal of R i.e there exists a positive integer n such that $P^n = 0$. Then I is an ideal of X. We show $I^n = 0$.
On the contrary, if possible let $I^n \neq 0$. So there exists nonzero elements in I^n.
Let $t \in I^n$ be a nonzero element.
Then $t = t_1\gamma_1t_2\gamma_2 \ldots\ldots \gamma_{n-1}t_n$ where $t_i \in I$ and $\gamma_i \in \Gamma$ and $t_i \neq 0, \gamma_i \neq 0$.
Since $t_i \in I \Rightarrow [\gamma_{i-1}, t_i] \in P$
So, $[\gamma_n, t_1][\gamma_1, t_2][\gamma_2, t_3] \ldots\ldots [\gamma_{n-1}, t_n] \in P^n$

$=> [\gamma_n, t_1\gamma_1 t_2\gamma_2 t_3 \ldots\ldots \gamma_{n-1} t_n] \in P^n => [\gamma_n, t] \in P^n = 0 => \gamma_n, t = 0$, which is a contradiction because $\gamma_n \neq 0$ and $t \neq 0$.
So, $I^n = 0$, i.e I is nilpotent ideal of X and hence the result.

Theorem 3.7: Every primary ideal of (X, Γ) defines a primary ideal of the right operator ring R and conversely.

Proof: Let I be a primary ideal of a gamma ring (X, Γ). Then,
$R' = \{\sum_i [\gamma_i, x_i] : \gamma_i \in \Gamma, x_i \in I\}$ is an ideal of the right operator ring R. We show R' is a primary ideal of R.
Let $x = \sum_i [\alpha_i, x_i]$, $y = \sum_j [\beta_j, y_j] \in R$ be any two elements such that $xy \in R'$
$=> \sum_i [\alpha_i, x_i] \sum_j [\beta_j, y_j] \in R' => \sum_{i,j} [\alpha_i, x_i\beta_j y_j] \in R' => x_i\beta_j y_j \in I$ for i, j
Since I is primary, so either $x_i \in I$ or $(y_j\beta_j)^{n-1} y_j \in I$ for some $n \in N$.
If $x_i \in I$ then $\sum_i [\alpha_i, x_i] \in R' => x \in R'$
And, if $(y_j\beta_j)^{n-1} y_j \in I$ then $\sum_j [\beta_j, (y_j\beta_j)^{n-1} y_j] \in R'$

$$=> \sum_j [\beta_j, y_j\beta_j y_j\beta_j \ldots\ldots \beta_j y_j] \in R'$$

$$=> \sum_j [\beta_j, y_j] \sum_j [\beta_j, y_j] \ldots\ldots \sum_j [\beta_j, y_j] \in R'$$

$$=> (\sum_j [\beta_j, y_j])^n \in R' => y^n \in R'$$

So, $xy \in R' => x \in R'$ or $y^n \in R'$. Thus R' is a primary ideal of R.
Conversely, let $P = \{\sum_i [\gamma_i, x_i] : \gamma_i \in \Gamma, x_i \in A\}$ be a primary ideal of R. Then A is an ideal of X. We show A is primary.
For this let $a, b \in X$ be two elements such that $a\gamma b \in A, \gamma \in \Gamma$
Then, $[\gamma, a\gamma b] \in P => [\gamma, a][\gamma, b] \in P$
Since P is primary ideal of R, so $[\gamma, a] \in P$ or $[\gamma, b]^n \in P$
If $[\gamma, a] \in P$ then $a \in A$
And, if $[\gamma, b]^n \in P$ then $[\gamma, b][\gamma, b] \ldots\ldots [\gamma, b] \in P$
$=> [\gamma, b\gamma b\gamma \ldots\ldots \gamma b] \in P => [\gamma, (b\gamma)^{n-1} b] \in P => (b\gamma)^{n-1} b \in A$
Thus $a\gamma b \in A => a \in A$ or $(b\gamma)^{n-1} b \in A$
So A is a primary ideal of X. Hence the result.

Theorem 3.8: Every semi-prime ideal of (X, Γ) defines a semi-prime ideal of the right operator ring R and conversely.

Proof: Let I be a semi-prime ideal of a gamma ring (X, Γ). Then,
$R' = \{\sum_i [\gamma_i, x_i] : \gamma_i \in \Gamma, x_i \in I\}$ is an ideal of the right operator ring R. We show R' is a semi-prime ideal of R.
$x = \sum_i [\gamma_i, x_i] \in R$ be any element such that $x^2 \in R'$

$$=> \sum_i [\gamma_i, x_i] \sum_i [\gamma_i, x_i] \in R'$$

$$=> \sum_i [\gamma_i, x_i\gamma_i x_i] \in R'$$

$=> x_i\gamma_i x_i \in I$ for all i
$=> x_i \in I$ for all i [Since I is semi-prime]

$$=> \sum_i [\gamma_i, x_i] \in R'$$

$=> x \in R'$
Thus we get, $x^2 \in R' => x \in R'$. So R' is semi-prime.
Conversely, let $P = \{\sum_i [\gamma_i, x_i] : \gamma_i \in \Gamma, x_i \in A\}$ be a semi-prime ideal of R. Then A is an ideal of X. We show A is semi-prime.
Let $a \in X$ be any element such that $a\gamma a \in A$ for $\gamma \in \Gamma$
$=> [\gamma, a\gamma a] \in P => [\gamma, a][\gamma, a] \in P => [\gamma, a]^2 \in P => [\gamma, a] \in P => a \in A$.
So A is semi-prime and hence the result.

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